

Rainbow connections of graphs – A survey*

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Abstract

The concept of rainbow connection was introduced by Chartrand et al. in 2008. It is fairly interesting and recently quite a lot papers have been published about it. In this survey we attempt to bring together most of the results and papers that dealt with it. We begin with an introduction, and then try to organize the work into five categories, including (strong) rainbow connection number, rainbow k -connectivity, k -rainbow index, rainbow vertex-connection number, algorithms and computational complexity. This survey also contains some conjectures, open problems or questions.

Keywords: rainbow path, (strong) rainbow connection number, rainbow k -connectivity, k -rainbow index, rainbow vertex-connection number, algorithm, computational complexity

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1 Introduction

1.1 Motivation and definitions

Connectivity is perhaps the most fundamental graph-theoretic subject, both in combinatorial sense and the algorithmic sense. There are many elegant and powerful results on connectivity in graph theory. There are also many ways to strengthen the connectivity concept, such as requiring hamiltonicity, k -connectivity, imposing bounds on the diameter, and so on. An interesting way to strengthen the connectivity requirement, the rainbow connection, was introduced by Chartrand, Johns, McKeon and Zhang [12] in 2008, which is restated as follows:

This new concept comes from the communication of information between agencies of government. The Department of Homeland Security of USA was created in 2003 in response

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to the weaknesses discovered in the transfer of classified information after the September 11, 2001 terrorist attacks. Ericksen [25] made the following observation: An unanticipated aftermath of those deadly attacks was the realization that law enforcement and intelligence agencies couldn't communicate with each other through their regular channels, from radio systems to databases. The technologies utilized were separate entities and prohibited shared access, meaning that there was no way for officers and agents to cross check information between various organizations.

While the information needs to be protected since it relates to national security, there must also be procedures that permit access between appropriate parties. This two-fold issue can be addressed by assigning information transfer paths between agencies which may have other agencies as intermediaries while requiring a large enough number of passwords and firewalls that is prohibitive to intruders, yet small enough to manage (that is, enough so that one or more paths between every pair of agencies have no password repeated). An immediate question arises: What is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies so that the passwords along each path are distinct?

This situation can be modeled by graph-theoretic model. Let G be a nontrivial connected graph on which an edge-coloring $c : E(G) \rightarrow \{1, 2, \dots, n\}$, $n \in \mathbb{N}$, is defined, where adjacent edges may be colored the same. A path is *rainbow* if no two edges of it are colored the same. An edge-coloring graph G is *rainbow connected* if any two vertices are connected by a rainbow path. An edge-coloring under which G is rainbow connected is called a *rainbow coloring*. Clearly, if a graph is rainbow connected, it must be connected. Conversely, any connected graph has a trivial edge-coloring that makes it rainbow connected; just color each edge with a distinct color. Thus, we define the *rainbow connection number* of a connected graph G , denoted by $rc(G)$, as the smallest number of colors that are needed in order to make G rainbow connected [12]. A rainbow coloring using $rc(G)$ colors is called a *minimum rainbow coloring*. So the question mentioned above can be modeled by means of computing the value of rainbow connection number. By definition, if H is a connected spanning subgraph of G , then $rc(G) \leq rc(H)$. For a basic introduction to the topic, we refer the readers to Chapter 11 in [16].

In addition to regarding as a natural combinatorial measure and its application for the secure transfer of classified information between agencies, rainbow connection number can also be motivated by its interesting interpretation in the area of networking [10]: Suppose that G represents a network (e.g., a cellular network). We wish to route messages between any two vertices in a pipeline, and require that each link on the route between the vertices (namely, each edge on the path) is assigned a distinct channel (e.g. a distinct frequency). Clearly, we want to minimize the number of distinct channels that we use in our network. This number is precisely $rc(G)$.

Let c be a rainbow coloring of a connected graph G . For any two vertices u and v of G , a *rainbow $u - v$ geodesic* in G is a rainbow $u - v$ path of length $d(u, v)$, where $d(u, v)$ is the distance between u and v in G . A graph G is *strong rainbow connected* if there exists a

rainbow $u-v$ geodesic for any two vertices u and v in G . In this case, the coloring c is called a *strong rainbow coloring* of G . Similarly, we define the *strong rainbow connection number* of a connected graph G , denoted $src(G)$, as the smallest number of colors that are needed in order to make G strong rainbow connected [12]. Note that this number is also called the *rainbow diameter number* in [10]. A strong rainbow coloring of G using $src(G)$ colors is called a *minimum strong rainbow coloring* of G . Clearly, we have $diam(G) \leq rc(G) \leq src(G) \leq m$, where $diam(G)$ denotes the diameter of G and m is the size of G .

In a rainbow coloring, we only need to find one rainbow path connecting any two vertices. So there is a natural generalization: the number of rainbow paths between any two vertices is at least an integer k with $k \geq 1$ in some edge-coloring. A well-known theorem of Whitney [55] shows that in every κ -connected graph G with $\kappa \geq 1$, there are k internally disjoint $u-v$ paths connecting any two distinct vertices u and v for every integer k with $1 \leq k \leq \kappa$. Similar to rainbow coloring, we call an edge-coloring a *rainbow k -coloring* if there are at least k internally disjoint $u-v$ paths connecting any two distinct vertices u and v . Chartrand, Johns, McKeon and Zhang [13] defined the *rainbow k -connectivity* $rc_k(G)$ of G to be the minimum integer j such that there exists a j -edge-coloring which is a *rainbow k -coloring*. A rainbow k -coloring using $rc_k(G)$ colors is called a *minimum rainbow k -coloring*. By definition, $rc_k(G)$ is the generalization of $rc(G)$ and $rc_1(G) = rc(G)$ is the rainbow connection number of G . By coloring the edges of G with distinct colors, we see that every two vertices of G are connected by k internally disjoint rainbow paths and that $rc_k(G)$ is defined for every $1 \leq k \leq \kappa$. So $rc_k(G)$ is well-defined. Furthermore, $rc_k(G) \leq rc_j(G)$ for $1 \leq k \leq j \leq \kappa$. Note that this new defined *rainbow k -connectivity* computes the number of *colors*, this is distinct with connectivity (edge-connectivity) which computes the number of internally (edge) disjoint *paths*. We can also call it *rainbow k -connection number*.

Now we introduce another generalization of rainbow connection number by Chartrand, Okamoto and Zhang [15]. Let G be an edge-colored nontrivial connected graph of order n . A tree T in G is a *rainbow tree* if no two edges of T are colored the same. Let k be a fixed integer with $2 \leq k \leq n$. An edge coloring of G is called a *k -rainbow coloring* if for every set S of k vertices of G , there exists a rainbow tree in G containing the vertices of S . The *k -rainbow index* $rx_k(G)$ of G is the minimum number of colors needed in a k -rainbow coloring of G . A k -rainbow coloring using $rx_k(G)$ colors is called a *minimum k -rainbow coloring*. Thus $rx_2(G)$ is the rainbow connection number $rc(G)$ of G . It follows, for every nontrivial connected graph G of order n , that $rx_2(G) \leq rx_3(G) \leq \dots \leq rx_n(G)$.

The above four new graph-parameters are all defined in edge-colored graphs. Krivelevich and Yuster [36] naturally introduced a new parameter corresponding to rainbow connection number which is defined on vertex-colored graphs. A vertex-colored graph G is *rainbow vertex-connected* if any two vertices are connected by a path whose *internal* vertices have distinct colors. A vertex-coloring under which G is rainbow vertex-connected is called a *rainbow vertex-coloring*. The *rainbow vertex-connection number* of a connected graph G , denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. The minimum rainbow vertex-coloring is defined sim-

ilarly. Obviously, we always have $rvc(G) \leq n - 2$ (except for the singleton graph), and $rvc(G) = 0$ if and only if G is a clique. Also clearly, $rvc(G) \geq diam(G) - 1$ with equality if the diameter of G is 1 or 2.

Note that $rvc(G)$ may be much smaller than $rc(G)$ for some graph G . For example, $rvc(K_{1,n-1}) = 1$ while $rc(K_{1,n-1}) = n - 1$. $rvc(G)$ may also be much larger than $rc(G)$ for some graph G . For example, take n vertex-disjoint triangles and, by designating a vertex from each of them, add a complete graph on the designated vertices. This graph has n cut-vertices and hence $rvc(G) \geq n$. In fact, $rvc(G) = n$ by coloring only the cut-vertices with distinct colors. On the other hand, it is not difficult to see that $rc(G) \leq 4$. Just color the edges of the K_n with, say, color 1, and color the edges of each triangle with the colors 2, 3, 4.

In Section 2, we will focus on the rainbow connection number and strong rainbow connection number. We collect many upper bounds for these two parameters. From Section 3 to Section 5, we survey on the other three parameters: rainbow k -connectivity, k -rainbow index, rainbow vertex-connection number, respectively. In the last section, we sum up the results on algorithms and computational complexity.

1.2 Terminology and notations

All graphs considered in this survey are finite, simple and undirected. We follow the notations and terminology of [7] for all those not defined here. We use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of G , respectively. For any subset X of $V(G)$, let $G[X]$ denote the subgraph induced by X , and $E[X]$ the edge set of $G[X]$; similarly, for any subset F of $E(G)$, let $G[F]$ denote the subgraph induced by F . Let \mathcal{G} be a set of graphs, then $V(\mathcal{G}) = \bigcup_{G \in \mathcal{G}} V(G)$, $E(\mathcal{G}) = \bigcup_{G \in \mathcal{G}} E(G)$. We define a *clique* in a graph G to be a complete subgraph of G , and a *maximal clique* is a clique that is not contained in any larger clique of G . For a set S , $|S|$ denotes the cardinality of S . An edge in a connected graph is called a *bridge*, if its removal disconnects the graph. A graph with no bridges is called a *bridgeless* graph. A vertex is called *pendant* if its degree is 1. We call a path of G with length k a *pendant k -length path* if one of its end vertex has degree 1 and all inner vertices have degree 2 in G . By definition, a pendant k -length path contains a pendant ℓ -length path ($1 \leq \ell \leq k$). A pendant 1-length path is a *pendant edge*. We denote C_n a cycle with n vertices. For $n \geq 3$, the *wheel* W_n is constructed by joining a new vertex to every vertex of C_n . We use $g(G)$ to denote the girth of G , that is, the length of a shortest cycle of G .

Let G be a connected graph. Recall that the distance between two vertices u and v in G , denoted by $d(u, v)$, is the length of a shortest path between them in G . The *eccentricity* of a vertex v is $ecc(v) := \max_{x \in V(G)} d(v, x)$. The *diameter* of G is $diam(G) := \max_{x \in V(G)} ecc(x)$. The *radius* of G is $rad(G) := \min_{x \in V(G)} ecc(x)$. Distance between a vertex v and a set $S \subseteq V(G)$ is $d(v, S) := \min_{x \in S} d(v, x)$. The *k -step open neighbourhood* of a set $S \subseteq V(G)$ is $N^k(S) := \{x \in V(G) | d(x, S) = k\}$, $k \in \{0, 1, 2, \dots\}$. A set $D \subseteq V(G)$ is called a *k -step dominating set* of G , if every vertex in G is at a distance at most k from D . Further, if D

induces a connected subgraph of G , it is called a *connected k -step dominating set* of G . The cardinality of a minimum connected k -step dominating set in G is called its *connected k -step domination number*, denoted by $\gamma_c^k(G)$. We call a two-step dominating set *k -strong* [36] if every vertex that is not dominated by it has at least k neighbors that are dominated by it. In [11], Chandran, Das, Rajendraprasad and Varma made two new definitions which will be useful in the sequel. A dominating set D in a graph G is called a *two-way dominating set* if every pendant vertex of G is included in D . In addition, if $G[D]$ is connected, we call D a *connected two-way dominating set*. A (connected) two-step dominating set D of vertices in a graph G is called a (connected) *two-way two-step dominating set* if (i) every pendant vertex of G is included in D and (ii) every vertex in $N^2(D)$ has at least two neighbours in $N^1(D)$. Note that if $\delta(G) \geq 2$, then every (connected) dominating set in G is a (connected) two-way dominating set.

A subgraph H of a graph G is called *isometric* if distance between any pair of vertices in H is the same as their distance in G . The size of a largest isometric cycle in G is denoted by $iso(G)$. A graph is called *chordal* if it contains no induced cycles of length greater than 3. The *chordality* of a graph G is the length of a largest induced cycle in G . Note that every isometric cycle is induced and hence $iso(G)$ is at most the chordality of G . For $k \leq \alpha(G)$, we use $\sigma_k(G)$ to denote the minimum degree sum that is taken over all independent sets of k vertices of G , where $\alpha(G)$ is the number of elements of a maximum independent set of G .

2 (Strong) Rainbow connection number

2.1 Basic results

In [12], Chartrand, Johns, McKeon and Zhang did some basic research on the (strong) rainbow connection numbers of graphs. They determined the precise (strong) rainbow connection numbers of several special graph classes including trees, complete graphs, cycles, wheel graphs, complete bipartite graphs and complete multipartite graphs.

Proposition 2.1 [12] *Let G be a nontrivial connected graph of size m . Then*

- (a) $rc(G) = 1$ if and only if G is complete, $src(G) = 1$ if and only if G is complete;
- (b) $rc(G) = 2$ if and only if $src(G) = 2$;
- (c) $rc(G) = m$ if and only if G is a tree, $src(G) = m$ if and only if G is a tree.

Proposition 2.2 [12] *For each integer $n \geq 4$, $rc(C_n) = src(C_n) = \lceil \frac{n}{2} \rceil$.*

Proposition 2.3 [12] *For each integer $n \geq 3$, we have*

$$rc(W_n) = \begin{cases} 1 & \text{if } n = 3, \\ 2 & \text{if } 4 \leq n \leq 6, \\ 3 & \text{if } n \geq 7. \end{cases}$$

and $src(W_n) = \lceil \frac{n}{3} \rceil$.

Proposition 2.4 [12] For integers s and t with $2 \leq s \leq t$, $rc(K_{s,t}) = \min\{\lceil \sqrt[s]{t} \rceil, 4\}$, and for integers s and t with $1 \leq s \leq t$, $src(K_{s,t}) = \lceil \sqrt[s]{t} \rceil$.

Proposition 2.5 [12] Let $G = K_{n_1, n_2, \dots, n_k}$ be a complete k -partite graph, where $k \geq 3$ and $n_1 \leq n_2 \leq \dots \leq n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$. Then

$$rc(G) = \begin{cases} 1 & \text{if } n_k = 1, \\ 2 & \text{if } n_k \geq 2 \text{ and } s > t, \\ \min\{\lceil \sqrt[s]{t} \rceil, 3\} & \text{if } s \leq t. \end{cases}$$

and

$$src(G) = \begin{cases} 1 & \text{if } n_k = 1, \\ 2 & \text{if } n_k \geq 2 \text{ and } s > t, \\ \lceil \sqrt[s]{t} \rceil & \text{if } s \leq t. \end{cases}$$

By Proposition 2.1, it follows that for every positive integer a and for every tree T of size a , $rc(T) = src(T) = a$. Furthermore, for $a \in \{1, 2\}$, $rc(G) = a$ if and only if $src(G) = a$. If $a = 3, b \geq 4$, then by Proposition 2.3, $rc(W_{3b}) = 3$ and $src(W_{3b}) = b$. For $a \geq 4$, we have the following.

Theorem 2.6 [12] Let a and b be positive integers with $a \geq 4$ and $b \geq \frac{5a-6}{3}$. Then there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$.

Then, combining Propositions 2.1 and 2.3 with Theorem 2.6, they got the following result.

Corollary 2.7 [12] Let a and b be positive integers. If $a = b$ or $3 \leq a < b$ and $b \geq \frac{5a-6}{3}$, then there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$.

Finally, they thought the question that whether the condition $b \geq \frac{5a-6}{3}$ can be deleted? and raised the following conjecture:

Conjecture 2.8 [12] Let a and b be positive integers. Then there exists a connected graph G such that $rc(G) = a$ and $src(G) = b$ if and only if $a = b \in \{1, 2\}$ or $3 \leq a \leq b$.

In [19], Chen and Li gave a confirmative solution to this conjecture by showing a class of graphs with given rainbow connection number a and strong rainbow connection number b .

From the above several propositions, we know $rc(G) = src(G)$ hold for some special graph classes. A difficult problem is following:

Problem 2.9 *Characterize graphs G for which $rc(G) = src(G)$, or, give some sufficient conditions to guarantee $rc(G) = src(G)$.*

Recall the fact that if H is a connected spanning subgraph of a nontrivial (connected) graph G , then $rc(G) \leq rc(H)$. This fact is very useful to bounding the value of $rc(G)$ by giving bounds for its connected spanning subgraphs. We have noted that if, in addition, $diam(H) = 2$, then $src(G) \leq src(H)$. The authors of [12] naturally raised the following conjecture:

Conjecture 2.10 [12] *If H is a connected spanning subgraph of a nontrivial (connected) graph G , then $src(G) \leq src(H)$.*

Recently, this conjecture was disproved by Chakraborty, Fischer, Matsliah and Yuster [10]. They showed the following example: see Figure 2.1, here G is obtained from H by adding the edge $e = uv$, then H is a connected spanning subgraph of G . It is easy to show that there is a strong rainbow coloring of H which costs six colors, but the graph G costs at least seven colors to ensure its strong rainbow connection.

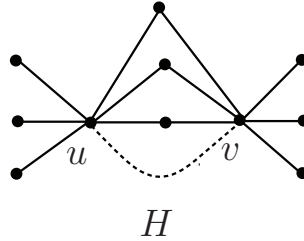


Figure 2.1 A counterexample to Conjecture 2.10.

Suppose that G contains two bridges $e = uv$ and $f = xy$. Then $G - e - f$ contains three components $G_i (1 \leq i \leq 3)$, where two of these components contain one of u, v, x and y and the third component contains two of these four vertices, say $u \in V(G_1)$, $x \in V(G_2)$ and $v, y \in V(G_3)$. If S is a set of k vertices contains u and x , then every tree whose vertex set contains S must also contain the edges e and f . This gives us a necessary condition for an edge-colored graph to be k -rainbow colored.

Observation 2.11 [15] *Let G be a connected graph of order n containing two bridges e and f . For each integer k with $2 \leq k \leq n$, every k -rainbow coloring of G must assign distinct colors to e and f .*

From Observation 2.11, we know that if G is rainbow connected under some edge-coloring, then any two bridges obtain distinct colors.

2.2 Upper bounds for rainbow connection number

We know that it is almost impossible to give the precise rainbow connection number of a given arbitrary graph, so we aim to give some nice bounds for it, especially sharp upper bounds.

In [9], Caro, Lev, Roditty, Tuza and Yuster investigated the extremal graph-theoretic behavior of rainbow connection number. Motivated by the fact that there are graphs with minimum degree 2 and with $rc(G) = n - 3$ (just take two vertex-disjoint triangles and connect them by a path of length $n - 5$), it is interesting to study the rainbow connection number of graphs with minimum degree at least 3 and they thought of the following question: is it true that minimum degree at least 3 guarantees $rc(G) \leq \alpha n$ where $\alpha < 1$ is independent of n ? This turns out to be true, and they proved:

Theorem 2.12 [9] *If G is a connected graph with n vertices and $\delta(G) \geq 3$, then $rc(G) < \frac{5}{6}n$.*

In the proof of Theorem 2.12, they first gave an upper bound for the rainbow connection number of 2-connected graphs (see Theorem 2.23), then from it, they next derived an upper bound for the rainbow connection number of connected bridgeless graphs (see Theorem 2.25).

The constant $5/6$ appearing in Theorem 2.12 is not optimal, but it probably cannot be replaced with a constant smaller than $\frac{3}{4}$, since there are 3-regular connected graphs with $rc(G) = diam(G) = \frac{3n-10}{4}$, and one of such graphs can be constructed as follows [53]: Take two vertex disjoint copies of the graph $K_5 - P_3$ and label the two vertices of degree 2 with w_1 and w_{2k+2} , where $k \geq 1$ is an integer. Next join w_1 and w_{2k+2} by a path of length $2k + 1$ and label the vertices with $w_1, w_2, \dots, w_{2k+2}$. Now for $1 \leq i \leq k$ every edge $w_{2i}w_{2i+1}$ is replaced by a $K_4 - e$ and we identify the two vertices of degree 2 in $K_4 - e$ with w_{2i} and w_{2i+1} . The resulting graph G_{4k+10} is 3-regular, has order $n = 4k + 10$ and $rc(G_{4k+10}) = diam(G_{4k+10}) = 3k + 5 = \frac{3n-10}{4}$. Then Caro, Lev, Roditty, Tuza and Yuster conjectured:

Conjecture 2.13 [9] *If G is a connected graph with n vertices and $\delta(G) \geq 3$, then $rc(G) < \frac{3}{4}n$.*

Schiermeyer proved the conjecture in [53] by showing the following result:

Theorem 2.14 [53] *If G is a connected graph with n vertices and $\delta(G) \geq 3$, then $rc(G) < \frac{3n-1}{4}$.*

For 2-connected graphs Theorem 2.14 is true by Theorem 2.23. Hence it remains to prove it for graphs with connectivity 1. Schiermeyer extended the concept of rainbow connection number as follows: Additionally we require that any two edges of G have different colors whenever they belong to different blocks of G . The corresponding rainbow connection

number will be denoted by $rc^*(G)$. Then they derived Theorem 2.14 by first proving the following result: let G be a connected graph with n vertices, connectivity 1, and $\delta \geq 3$, then $rc^*(G) \leq \frac{3n-10}{4}$.

Not surprisingly, as the minimum degree increases, the graph would become more dense and therefore the rainbow connection number would decrease. Specifically, Caro, Lev, Roditty, Tuza and Yuster also proved the following upper bounds in term of minimum degree.

Theorem 2.15 [9] *If G is a connected graph with n vertices and minimum degree δ , then*

$$rc(G) \leq \min\left\{n\frac{\ln \delta}{\delta}(1 + o_\delta(1)), n\frac{4 \ln \delta + 3}{\delta}\right\}.$$

In the proof, they used the concept of a connected two-dominating set (A set of vertices S of G is called a *connected two-dominating set* if S induces a connected subgraph of G and, furthermore, each vertex outside of S has at least two neighbours in S) and the probabilistic method. They showed that in any case it cannot be improved below $\frac{3n}{\delta+1} - \frac{\delta+7}{\delta+1}$ as they constructed a connected n -vertex graph with minimum degree δ and this diameter: Take m copies of $K_{\delta+1}$, denoted by X_1, \dots, X_m and label the vertices of X_i with $x_{i,1}, \dots, x_{i,\delta+1}$. Take two copies of $K_{\delta+2}$, denoted by X_0, X_{m+1} and similarly label their vertices. Now, connect $x_{i,2}$ with $x_{i+1,1}$ for $i = 0, \dots, m$ with an edge, and delete the edges $(x_{i,1}, x_{i,2})$ for $i = 0, \dots, m+1$. The obtained graph has $n = (m+2)(\delta+1) + 2$ vertices, and minimum degree δ (and maximum degree $\delta+1$). It is straightforward to verify that a shortest path from $x_{0,1}$ to $x_{m+1,2}$ has length $3m+5 = \frac{3n}{\delta+1} - \frac{\delta+7}{\delta+1}$.

This, naturally, raised the open problem of determining the true behavior of $rc(G)$ as a function of δ .

In [10], Chakraborty, Fischer, Matsliah and Yuster proved that any connected n -vertex graph with minimum degree $\Theta(n)$ has a bounded rainbow connection.

Theorem 2.16 [10] *For every $\epsilon > 0$ there is a constant $C = C(\epsilon)$ such that if G is a connected graph with n vertices and minimum degree at least ϵn , then $rc(G) \leq C$.*

The proof of Theorem 2.16 is based upon a modified degree-form version of Szemerédi Regularity Lemma (see [35] for a good survey on Regularity Lemma) that they proved.

The above lower bound construction suggests that the logarithmic factor in their upper bound may not be necessary and that, in fact $rc(G) \leq Cn/\delta$ where C is a universal constant. If true, notice that for graphs with a linear minimum degree ϵn , this implies that $rc(G)$ is at most C/ϵ . However, Theorem 2.16 does not even guarantee the weaker claim that $rc(G)$ is a constant. The constant $C = C(\epsilon)$ they obtained is a *tower function* in $1/\epsilon$ and in particular extremely far from being reciprocal to $1/\epsilon$.

Finally, Krivelevich and Yuster in [36] determined the behavior of $rc(G)$ as a function of $\delta(G)$ and resolved the above-mentioned open problem.

Theorem 2.17 [36] *A connected graph G with n vertices has $rc(G) < \frac{20n}{\delta(G)}$.*

The proof of Theorem 2.17 uses the concept of connected two-step dominating set. Krivelevich and Yuster first proved that for a connected graph H with minimum degree k and n vertices, there exists a two-step dominating set S whose size is at most $\frac{n}{k+1}$ and there is a connected two-step dominating set S' containing S with $|S'| \leq 5|S| - 4$. They found two edge-disjoint spanning subgraphs in a graph G with minimum degree at least $\lfloor \frac{\delta-1}{2} \rfloor$. Then they derived a rainbow coloring for G by giving a rainbow coloring to each subgraphs according to its connected two-step dominating set.

The authors noted that the constant 20 obtained by their proof is not optimal and can be slightly improved with additional effort. However, from the example below Theorem 2.15, one cannot expect to replace C by a constant smaller than 3.

Motivated by the results of Theorems 2.14, 2.15 and 2.17, Schiermeyer raised the following open problem in [53].

Problem 2.18 [53] *For every $k \geq 2$ find a minimal constant c_k with $0 < c_k \leq 1$ such that $rc(G) \leq c_k n$ for all graphs G with minimum degree $\delta(G) \geq k$. Is it true that $c_k = \frac{3}{k+1}$ for all $k \geq 2$?*

This is true for $k = 2, 3$ as shown before ($c_2 = 1$ and $c_3 = \frac{3}{4}$).

Recently, Chandran, Das, Rajendraprasad and Varma [11] nearly settled the above problem. They used the concept of a connected two-way two-step dominating set in the argument and they first proved the following result.

Theorem 2.19 [11] *If D is a connected two-way two-step dominating set in a graph G , then $rc(G) \leq rc(G[D]) + 6$.*

Furthermore, they gave a nearly sharp bound for the size of D by showing that every connected graph G of order $n \geq 4$ and minimum degree δ has a connected two-way two-step dominating set D of size at most $\frac{3n}{\delta+1} - 2$; moreover, for every $\delta \geq 2$, there exist infinitely many connected graphs G such that $\gamma_c^2(G) \geq \frac{3(n-2)}{\delta+1} - 4$. Then the following result is easy.

Theorem 2.20 [11] *For every connected graph G of order n and minimum degree δ ,*

$$rc(G) \leq \frac{3n}{\delta+1} + 3.$$

Moreover, for every $\delta \geq 2$, there exist infinitely many connected graphs G such that $rc(G) \geq \frac{3(n-2)}{\delta+1} - 1$.

Theorem 2.20 answers Problem 2.18 in the affirmative but up to an additive constant of 3. Moreover, this bound is seen to be tight up to additive factors by the construction

mentioned in [9] (see the example below Theorem 2.15) and [23]. And therefore, for graphs with linear minimum degree ϵn , the rainbow connection number is bounded by a constant.

Recently, Dong and Li [22] derived an upper bound on rainbow connection numbers of graphs under given degree sum condition σ_2 . Recall that for a graph G , $\sigma_2(G) = \min\{d(u) + d(v) \mid u, v \text{ are independent in } G\}$. Clearly, the degree sum condition σ_2 is weaker than the minimum degree condition.

Theorem 2.21 [22] *For a connected graph G of order n , $rc(G) \leq 6\frac{n-2}{\sigma_2+2} + 7$.*

Similar to the method of Theorem 2.20, they derived that every connected graph G of order n with at most one pendant vertex has a connected two-way two-step dominating set D of size at most $6\frac{n-2}{\sigma_2+2} + 2$. Then by using Theorem 2.19, they got the theorem.

From the example below Theorem 2.15, we know their bound are seen to be tight up to additive factors. Note that by the definition of σ_2 , we know $\sigma_2 \geq 2\delta$, so from Theorem 2.21, we can derive $rc(G) \leq 6\frac{n-2}{\sigma_2+2} + 7 \leq \frac{3(n-2)}{\delta+1} + 7$. And the bound in Theorem 2.21 can be seen as an improvement of that in Theorem 2.20.

With respect to the relation between $rc(G)$ and the connectivity $\kappa(G)$, mentioned in [53], Broersma asked a question at the IWOCA workshop:

Problem 2.22 [53] *What happens with the value $rc(G)$ for graphs with higher connectivity?*

For $\kappa(G) = 1$, Theorem 2.14 means that if G is a graph of order n , connectivity $\kappa(G) = 1$ and $\delta \geq 3$. Then $rc(G) \leq \frac{3n-1}{4}$. For $\kappa(G) = 2$, in the proof of Theorem 2.12, as we mentioned above, Caro, Lev, Roditty, Tuza and Yuster derived:

Theorem 2.23 [9] *If G is a 2-connected graph with n vertices then $rc(G) \leq \frac{2n}{3}$.*

That is, if G is a graph of order n , connectivity $\kappa(G) = 2$. Then $rc(G) \leq \frac{2n}{3}$.

From Theorem 2.20, we can easily obtain an upper bound of the rainbow connection number according to the connectivity:

$$rc(G) \leq \frac{3n}{\delta+1} + 3 \leq \frac{3n}{\kappa+1} + 3.$$

Therefore, for $\kappa(G) = 3$, $rc(G) \leq \frac{3n}{4} + 3$; for $\kappa(G) = 4$, $rc(G) \leq \frac{3n}{5} + 3$. Motivated by the results in [9], and by using the Fan Lemma, Li and Shi [41] improved this bound by showing the following result.

Theorem 2.24 ([41]) *If G is a 3-connected graph with n vertices, then $rc(G) \leq \frac{3(n+1)}{5}$.*

However, for general connectivity, there is no upper bound which is better than $\frac{3n}{\kappa+1} + 3$.

The following result is an important ingredient in the proof of Theorem 2.12 in [9].

Theorem 2.25 [9] *If G is a connected bridgeless graph with n vertices, then $rc(G) \leq \frac{4n}{5} - 1$.*

From Theorem 2.20, we can also easily obtain an upper bound of the rainbow connection number according to the edge-connectivity λ :

$$rc(G) \leq \frac{3n}{\delta + 1} + 3 \leq \frac{3n}{\lambda + 1} + 3.$$

Note that all the above upper bounds are determined by n and other parameters such as (edge)-connectivity, minimum degree. Diameter of a graph, and hence its radius, are obvious lower bounds for rainbow connection number. Hence it is interesting to see if there is an upper bound which is a function of the radius r or diameter alone. Such upper bounds were shown for some special graph classes in [11] which we will introduce in the sequel. But, for a general graph, the rainbow connection number cannot be upper bounded by a function of r alone. For instance, the star $K_{1,n}$ has a radius 1 but rainbow connection number n . Still, the question of whether such an upper bound exists for graphs with higher connectivity remains. Basavaraju, Chandran, Rajendraprasad and Ramaswamy [4] answered this question in the affirmative. The key of their argument is the following lemma, and in the proof of this lemma, we can obtain a connected $(k - 1)$ -step dominating set from a connected k -step dominating set.

Lemma 2.26 [4] *If G is a bridgeless graph, then for every connected k -step dominating set D^k of G , $k \geq 1$, there exists a connected $(k - 1)$ -step dominating set $D^{k-1} \supset D^k$ such that*

$$rc(G[D^{k-1}]) \leq rc(G[D^k]) + \min\{2k + 1, \zeta\},$$

where $\zeta = iso(G)$.

Given a graph G and a set $D \subset V(G)$, A D -ear is a path $P = (x_0, x_1, \dots, x_p)$ in G such that $P \cap D = \{x_0, x_p\}$. P may be a closed path, in which case $x_0 = x_p$. Further, P is called an *acceptable* D -ear if either P is a shortest D -ear containing (x_0, x_1) or P is a shortest D -ear containing (x_{p-1}, x_p) . Let $\mathcal{A} = \{a_1, a_2, \dots\}$ and $\mathcal{B} = \{b_1, b_2, \dots\}$ be two pools of colors, none of which are used to color $G[D^k]$. A D^k -ear $P = (x_0, x_1, \dots, x_p)$ will be called *evenly colored* if its edges are colored $a_1, a_2, \dots, a_{\lceil \frac{p}{2} \rceil}, b_{\lfloor \frac{p}{2} \rfloor}, \dots, b_2, b_1$ in that order. Basavaraju, Chandran, Rajendraprasad and Ramaswamy proved this lemma by constructing a sequence of sets $D^k = D_0 \subset D_1 \subset \dots \subset D_t = D^{k-1}$ and coloring the new edges in every induced graph $G[D_i]$ such that the following property is maintained for all $0 \leq i \leq t$: every $x \in D_i \setminus D^k$ lies in an evenly colored acceptable D^k -ear in $G[D_i]$.

The following theorem can be derived from Lemma 2.26 easily.

Theorem 2.27 [4] *For every connected bridgeless graph G ,*

$$rc(G) \leq \sum_{i=1}^r \min\{2i + 1, \zeta\} \leq r\zeta,$$

where r is the radius of G .

Theorem 2.27 has two corollaries.

Corollary 2.28 [4] *For every connected bridgeless graph G with radius r ,*

$$rc(G) \leq r(r+2).$$

Moreover, for every integer $r \geq 1$, there exists a bridgeless graph with radius r and $rc(G) = r(r+2)$.

Corollary 2.29 [4] *For every connected bridgeless graph G with radius r and chordality k ,*

$$rc(G) \leq \sum_{i=1}^r \min\{2i+1, k\} \leq rk.$$

Moreover, for every two integers $r \geq 1$ and $3 \leq k \leq 2r+1$, there exists a bridgeless graph G with radius r and chordality k such that $rc(G) = \sum_{i=1}^r \min\{2i+1, k\}$.

Corollary 2.28 answered the above question in the affirmative, the bound is sharp and is a function of the radius r alone. Basavaraju, Chandran, Rajendraprasad and Ramaswamy also demonstrated that the bound cannot be improved even if we assume stronger connectivity by constructing a κ -vertex-connected graph of radius r whose rainbow connection number is $r(r+2)$ for any two given integers $\kappa, r \geq 1$: Let $s(0) := 0, s(i) := 2\sum_{j=r}^{r-i+1} j$ for $1 \leq i \leq r$ and $t := s(r) = r(r+1)$. Let $V = V_0 \uplus V_1 \uplus \dots \uplus V_t$ where $V_i = \{x_{i,0}, x_{i,1}, \dots, x_{i,\kappa-1}\}$ for $0 \leq i \leq t-1$ and $V_t = \{x_{t,0}\}$. Construct a graph $X_{r,\kappa}$ on V by adding the following edges. $E(X) = \{\{x_{i,j}, x_{i',j'}\} : |i-i'| \leq 1\} \cup \{\{x_{s(i),0}, x_{s(i+1),0}\} : 0 \leq i \leq r-1\}$.

Corollary 2.29 generalises a result from [11] that the rainbow connection number of any bridgeless chordal graph is at most three times its radius as the chordality of a chordal graph is three.

In [9], Caro, Lev, Roditty, Tuza and Yuster also derived a result which gives an upper bound for rainbow connection number according to the order and the number of vertex-disjoint cycles. Here $\chi'(G)$ is the chromatic index of G .

Theorem 2.30 [9] *Suppose G is a connected graph with n vertices, and assume that there is a set of vertex-disjoint cycles that cover all but s vertices of G . Then $rc(G) < 3n/4 + s/4 - 1/2$. In particular:*

- (i). *If G has a 2-factor then $rc(G) < 3n/4$.*
- (ii). *If G is k -regular and k is even then $rc(G) < 3n/4$.*
- (iii). *If G is k -regular and $\chi'(G) = k$ then $rc(G) < 3n/4$.*

Another approach for achieving upper bounds is based on the size (number of edges) m of the graph. Those type of sufficient conditions are known as Erdős-Gallai type results. Research on the following Erdős-Gallai type problem has been started in [34].

Problem 2.31 [34] *For every k , $1 \leq k \leq n-1$, compute and minimize the function $f(n, k)$ with the following property: If $|E(G)| \geq f(n, k)$, then $rc(G) \leq k$.*

In [34], Kemnitz and Schiermeyer gave a lower bound for $f(n, k)$, i.e., $f(n, k) \geq \binom{n-k+1}{2} + (k-1)$. They also computed $f(n, k)$ for $k \in \{1, n-2, n-1\}$, i.e., $f(n, 1) = \binom{n}{2}$, $f(n, n-1) = n-1$, $f(n, n-2) = n$, and obtained $f(n, 2) = \binom{n-1}{2} + 1$ for $k = 2$.

In [48], Li and Sun provided a new approach to investigate the rainbow connection number of a graph G according to some constraints to its complement graph \overline{G} . They gave two sufficient conditions to guarantee that $rc(G)$ is bounded by a constant. By using the fact that $rc(G) \leq rc(H)$ where H is a connected spanning subgraph of a connected graph G , and the structure of its complement graph as well as Propositions 2.4 and 2.5, they derived the following result.

Theorem 2.32 [48] *For a connected graph G , if \overline{G} does not belong to the following two cases: (i) $\text{diam}(\overline{G}) = 2, 3$, (ii) \overline{G} contains exactly two connected components and one of them is trivial, then $rc(G) \leq 4$. Furthermore, this bound is best possible.*

For the remaining cases, $rc(G)$ can be very large as discussed in [48]. So They add a constraint: let \overline{G} be triangle-free, then G is claw-free. And they derived the following result. In their argument, Theorem 2.40 is useful.

Theorem 2.33 [48] *For a connected graph G , if \overline{G} is triangle-free, then $rc(G) \leq 6$.*

The readers may consider the rainbow connection number of a graph G according to some other condition to its complement graph.

Chen, Li and Lian [17] investigated Nordhaus-Gaddum-type result. A Nordhaus-Gaddum-type result is a (sharp) lower or upper bound on the sum or product of the values of a parameter for a graph and its complement. The name “Nordhaus-Gaddum-type” is so given because it is Nordhaus and Gaddum [49] who first established the following type of inequalities for chromatic number of graphs in 1956.

Theorem 2.34 [17] *Let G and \overline{G} be connected with $n \geq 4$, then*

$$4 \leq rc(G) + rc(\overline{G}) \leq n + 2.$$

Furthermore, the upper bound is sharp for $n \geq 4$ and the low bound is sharp for $n \geq 8$.

They also proved that $rc(G) + rc(\overline{G}) \geq 6$ for $n = 4, 5$; and $rc(G) + rc(\overline{G}) \geq 5$ for $n = 6, 7$ and these bounds are best possible.

2.3 For some graph classes

Some graph classes, such as line graphs, have many special properties, and by these properties we can get some interesting results on their rainbow connection numbers in terms of some graph parameters. For example, in [9] Caro, Lev, Roditty, Tuza and Yuster derived Theorem 2.23 according to the ear-decomposition of a 2-connected graph. In this subsection, we will introduce some results on rainbow connection numbers of line graphs, etc.

In [42] and [43], Li and Sun studied the rainbow connection numbers of line graphs in the light of particular properties of line graphs shown in [30] and [31]. They gave two sharp upper bounds for rainbow connection number of a line graph and one sharp upper bound for rainbow connection number of an iterated line graph.

Recall the *line graph* of a graph G is the graph $L(G)$ (or $L^1(G)$) whose vertex set $V(L(G)) = E(G)$, and two vertices e_1, e_2 of $L(G)$ are adjacent if and only if they are adjacent in G . The *iterated line graph* of a graph G , denoted by $L^2(G)$, is the line graph of the graph $L(G)$. More generally, the *k-iterated line graph* $L^k(G)$ is the line graph of $L^{k-1}(G)$ ($k \geq 2$). We also need the following new terminology.

For a connected graph G , we call G a *clique-tree-structure*, if it satisfies the following condition: each block is a maximal clique. We call a graph H a *clique-forest-structure*, if H is a disjoint union of some clique-tree-structures, that is, each component of a clique-forest-structure is a clique-tree-structure. By the above condition, we know that any two maximal cliques of G have at most one common vertex. Furthermore, G is formed by its maximal cliques. The *size* of a clique-tree(forest)-structure is the number of its maximal cliques. An example of clique-forest-structure is shown in Figure 2.2. If each block of a

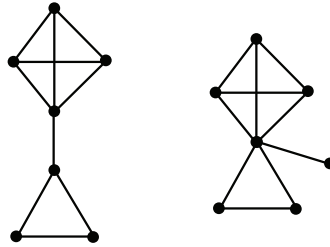


Figure 2.2 A clique-forest-structure with size 6 and 2 components.

clique-tree-structure is a triangle, we call it a *triangle-tree-structure*. Let ℓ be the size of a triangle-tree-structure. Then, by definition, it is easy to show that there are $2\ell + 1$ vertices in it. Similarly, we can give the definition of a *triangle-forest-structure* and there are $2\ell + c$ vertices in a triangle-forest-structure with size ℓ and c components. We denote n_2 the number of inner vertices (degrees at least 2) of a graph.

Theorem 2.35 [43] *For any set \mathcal{T} of t edge-disjoint triangles of a connected graph G , if*

the subgraph induced by the edge set $E(\mathcal{T})$ is a triangle-forest-structure, then

$$rc(L(G)) \leq n_2 - t.$$

Moreover, the bound is sharp.

Theorem 2.36 [43] *If G is a connected graph, \mathcal{T} is a set of t edge-disjoint triangles that cover all but n'_2 inner vertices of G and c is the number of components of the subgraph $G[E(\mathcal{T})]$, then*

$$rc(L(G)) \leq t + n'_2 + c.$$

Moreover, the bound is sharp.

Theorem 2.37 [43] *Let G be a connected graph with m edges and m_1 pendant 2-length paths. Then*

$$rc(L^2(G)) \leq m - m_1.$$

The equality holds if and only if G is a path of length at least 3.

In the proofs of the above three theorems, Li and Sun used the particular structure of line graphs and the observation: If G is a connected graph and $\{E_i\}_{i \in [t]}$ is a partition of the edge set of G into connected subgraphs $G_i = G[E_i]$, then $rc(G) \leq \sum_{i=1}^t rc(G_i)$ (see [42]).

The above three theorems give upper bounds for rainbow connection number of $L^k(G)$ ($k = 1, 2$) according to some parameters of G . One may consider the relation between $rc(G)$ and $rc(L(G))$.

Problem 2.38 *Determine the relationship between $rc(G)$ and $rc(L(G))$, is there an upper bound for one of these parameters in terms of the other?*

One also can consider the rainbow connection number of the general iterated line graph $L^k(G)$ when k is sufficiently large.

Problem 2.39 *Consider the value of $rc(L^k(G))$ as $k \rightarrow \infty$, is it bounded by a constant? or, does it convergence to a function of some graph parameters, such as the order n of G ? ■*

For Problem 2.39, we know if G is a cycle C_n ($n \geq 4$), then $L^k(G) = G$, so $rc(L^k(G)) = \lceil \frac{n}{2} \rceil$. But for many graphs, we know, as k grows, $L^k(G)$ will become more dense, and $rc(L^k(G))$ may decrease.

An *intersection graph* of a family of sets \mathcal{F} , is a graph whose vertices can be mapped to the sets in \mathcal{F} such that there is an edge between two vertices in the graph if and only if the corresponding two sets in \mathcal{F} have a non-empty intersection. An *interval graph* is an intersection graph of intervals on the real line. A *unit interval graph* is an intersection

graph of unit length intervals on the real line. A *circular arc graph* is an intersection graph of arcs on a circle. An independent triple of vertices x, y, z in a graph G is an *asteroidal triple* (AT), if between every pair of vertices in the triple, there is a path that does not contain any neighbour of the third. A graph without asteroidal triples is called an *AT-free graph* [20]. A graph G is a *threshold graph*, if there exists a weight function $w : V(G) \rightarrow \mathbb{R}$ and a real constant t such that two vertices $u, v \in V(G)$ are adjacent if and only if $w(u) + w(v) \geq t$. A bipartite graph $G(A, B)$ is called a *chain graph* if the vertices of A can be ordered as $A = (a_1, a_2, \dots, a_k)$ such that $N(a_1) \subseteq N(a_2) \subseteq \dots \subseteq N(a_k)$ [56]. In [11], Chandran, Das, Rajendraprasad and Varma investigated the rainbow connection numbers of these special graph classes. They first showed a result concerning the connected two-way dominating sets.

Theorem 2.40 [11] *If D is a connected two-way dominating set in a graph G , then*

$$rc(G) \leq rc(G[D]) + 3.$$

They also proved that every connected graph G of order n and minimum degree δ has a connected two-step dominating set D of size at most $\frac{3(n - |N^2(D)|)}{\delta + 1} - 2$.

From Theorem 2.40, the following result can be derived.

Theorem 2.41 [11] *Let G be a connected graph with $\delta(G) \geq 2$. Then*

- (i) *if G is an interval graph, $\text{diam}(G) \leq rc(G) \leq \text{diam}(G) + 1$, in particular, if G is a unit interval graph, then $rc(G) = \text{diam}(G)$;*
- (ii) *if G is AT-free, $\text{diam}(G) \leq rc(G) \leq \text{diam}(G) + 3$;*
- (iii) *if G is a threshold graph, $\text{diam}(G) \leq rc(G) \leq 3$;*
- (iv) *if G is a chain graph, $\text{diam}(G) \leq rc(G) \leq 4$;*
- (v) *if G is a circular arc graph, $\text{diam}(G) \leq rc(G) \leq \text{diam}(G) + 4$.*

Moreover, there exist threshold graphs and chain graphs with minimum degree at least 2 and rainbow connection number equal to the corresponding upper bound above. There exists an AT-free graph G with minimum degree at least 2 and $rc(G) = \text{diam}(G) + 2$, which is 1 less than the upper bound above.

Recall that the concept of rainbow connection number is of great use in transferring information of high security in multicomputer networks. Cayley graphs are very good models that have been used in communication networks. So, it is of significance to study the rainbow connection numbers of Cayley graphs. Li, Li and Liu [38] investigated the rainbow connection numbers of cayley graphs on Abelian groups.

Let Γ be a group, and let $a \in \Gamma$ be an element. We use $\langle a \rangle$ to denote the cyclic subgroup of Γ generated by a . The number of elements of $\langle a \rangle$ is called the *order* of a , denoted by $|a|$. A pair of elements a and b in a group commutes if $ab = ba$. A group is *Abelian* if every pair of its elements commutes. A *Cayley graph* of Γ with respect to S is the graph $C(\Gamma, S)$ with vertex set Γ in which two vertices x and y are adjacent if and only if $xy^{-1} \in S$ (or equivalently, $yx^{-1} \in S$), where $S \subseteq \Gamma \setminus \{1\}$ is closed under taking inverse [52].

Theorem 2.42 [38] *Given an Abelian group Γ and an inverse closed set $S \subseteq \Gamma \setminus \{1\}$, we have the following results:*

- (i) $rc(C(\Gamma, S)) \leq \min\{\sum_{a \in S^*} \lceil |a|/2 \rceil \mid S^* \subseteq S \text{ is a minimal generating set of } \Gamma\}$.
- (ii) *If S is an inverse closed minimal generating set of Γ , then*

$$\sum_{a \in S^*} \lceil |a|/2 \rceil \leq rc(C(\Gamma, S)) \leq src(C(\Gamma, S)) \leq \sum_{a \in S^*} \lceil |a|/2 \rceil,$$

where $S^* \subseteq S$ is a minimal generating set of Γ .

Moreover, if every element $a \in S$ has an even order, then

$$rc(C(\Gamma, S)) = src(C(\Gamma, S)) = \sum_{a \in S^*} |a|/2.$$

They also investigated the rainbow connection numbers of recursive circulants (see [50] for an introduction to recursive circulants).

Let G be an r -regular graph with n vertices. G is said to be *strongly regular* and denoted by $SRG(v, k, \lambda, \mu)$ if there are also integers λ and μ such that every two adjacent vertices have λ common neighbours and every two nonadjacent vertices have μ common neighbours. Clearly, a strongly regular graph with parameters (v, k, λ, μ) is connected if and only if $\mu \geq 1$. In [1], Ahadi and Dehghan derived the following result: For every connected strongly regular graph G , $rc(G) \leq 600$. As each strongly regular graph is a graph with $diam(G) = 2$, from our next subsection (Theorem 2.51), we know $rc(G) \leq 5$ if G is a strongly regular graph with parameters (v, k, λ, μ) , other than a star [39]. But 5 may not be the optimal upper bound, so one may consider the following question.

Question 2.43 [1] *Determine $\max\{rc(G) \mid G \text{ is an } SRG\}$.*

There are other results on some special graph classes. In [14], Chartrand, Johns, McKeon and Zhang investigated the rainbow connection numbers of cages, and in [33], Johns, Okamoto and Zhang investigated the rainbow connection numbers of small cubic graphs. The details are omitted.

2.4 For dense and sparse graphs

For any given graph G , we know $1 \leq rc(G) \leq src(G) \leq m$. Here a graph G is called a *dense graph* if its (strong) rainbow connection number is small, especially it is close to 1; while G is called a *sparse graph* if its (strong) rainbow connection number is large, especially it is close to m . By Proposition 2.1, the cases that $rc(G) = 1, src(G) = 1$ and $rc(G) = m, src(G) = m$ are clear. So we want to investigate other cases.

In [9], Caro, Lev, Roditty, Tuza and Yuster investigated the graphs with small rainbow connection numbers, and they gave a sufficient condition that guarantees $rc(G) = 2$.

Theorem 2.44 [9] *Any non-complete graph with $\delta(G) \geq n/2 + \log n$ has $rc(G) = 2$.*

We know that having diameter 2 is a necessary requirement for having $rc(G) = 2$, although certainly not sufficient (e.g., consider a star). Clearly, if $\delta(G) \geq n/2$ then $\text{diam}(G) = 2$, but we do not know if this guarantees $rc(G) = 2$. The above theorem shows that by slightly increasing the minimum degree assumption, $rc(G) = 2$ follows.

Kemnitz and Schiermeyer [34] gave a sufficient condition to guarantee $rc(G) = 2$ according to the number of edges m .

Theorem 2.45 [34] *Let G be a connected graph of order n and size m . If $\binom{n-1}{2} + 1 \leq m \leq \binom{n}{2} - 1$, then $rc(G) = 2$.*

Let $G = G(n, p)$ denote, as usual [3], the random graph with n vertices and edge probability p . For a graph property A and for a function $p = p(n)$, we say that $G(n, p)$ satisfies A *almost surely* if the probability that $G(n, p(n))$ satisfies A tends to 1 as n tends to infinity. We say that a function $f(n)$ is a *sharp threshold function* for the property A if there are two positive constants c and C so that $G(n, cf(n))$ almost surely does not satisfy A and $G(n, p)$ satisfies A almost surely for all $p \geq Cf(n)$. It is well known that all monotone graph properties have a sharp threshold function (see [6] and [27]). Since having $rc(G) \leq 2$ is a monotone graph property (adding edges does not destroy this property), it has a sharp threshold function. The following theorem establishes it.

Theorem 2.46 [9] *$p = \sqrt{\log n/n}$ is a sharp threshold function for the graph property $rc(G(n, p)) \leq 2$.*

Theorem 2.44 asserts that minimum degree $n/2 + \log n$ guarantees $rc(G) = 2$. Clearly, minimum degree $n/2 - 1$ does not, as there are connected graphs with minimum degree $n/2 - 1$ and diameter 3 (just take two vertex-disjoint cliques of order $n/2$ each and connect them by a single edge. It is therefore interesting to raise:

Problem 2.47 [9] *Determine the minimum degree threshold that guarantees $rc(G) = 2$.*

By Proposition 2.1, we know that the problem of considering graphs with $rc(G) = 2$ is equivalent to that of considering graphs with $\text{src}(G) = 2$.

A bipartite graph which is not complete has diameter at least 3. A proof similar to that of Theorem 2.46 gives the following result.

Theorem 2.48 [9] *Let $c = 1/\log(9/7)$. If G is a non-complete bipartite graph with n vertices and any two vertices in the same vertex class have at least $2c\log n$ common neighbors in the other vertex class, then $rc(G) = 3$.*

The following theorem asserts however that having diameter 2 and only logarithmic minimum degree suffices to guarantee rainbow connection 3.

Theorem 2.49 [10] *If G is an n -vertex graph with diameter 2 and minimum degree at least $8\log n$, then $rc(G) \leq 3$.*

Since a graph with minimum degree $n/2$ is connected and has diameter 2, we have as an immediate result [10]: If G is an n -vertex graph with minimum degree at least $n/2$ then $rc(G) \leq 3$. We know that any graph G with $rc(G) = 2$ must have $diam(G) = 2$, so graphs with $rc(G) = 2$ belong to the graph class with $diam(G) = 2$. By Corollary 2.28, we know that for a bridgeless graph with $diam(G) = 2$, $rc(G) \leq r(r+2) \leq 8$. As $rc(G)$ is at least the number of bridges of G , so it may be very large if the number of bridges of G is sufficiently large by Observation 2.11. So there is an interesting problem:

Problem 2.50 *For any bridgeless graph G with $diam(G) = 2$, determine the smallest constant c such that $rc(G) \leq c$.*

Recently, Li, Li and Liu [39] derived that $c \leq 5$ by showing the following result:

Theorem 2.51 [39] *$rc(G) \leq 5$ if G is a bridgeless graph with diameter 2; and that $rc(G) \leq k + 2$ if G is a connected graph with diameter 2 and k bridges, where $k \geq 1$.*

In the proof, Li, Li and Liu derived that if G is a bridgeless graph with order n and diameter 2, then it is either 2-connected, or it has only one cut vertex v , furthermore, v is the center of G with radius 1. They showed that 5 is almost best possible as there are infinity many bridgeless graphs with diameter 2 whose rainbow connection numbers are 4, however they have not found examples of such graphs with $rc(G) = 5$. The bound $k + 2$ is sharp as there are infinity graphs with diameter 2 and k bridges whose rainbow connection numbers attain this bound [39].

In [46] and [47], Li and Sun investigated the graphs with large rainbow connection numbers and strong rainbow connection numbers, respectively. They derived the two following results. Note that each path P_j in the member of graph class \mathcal{G}_i ($1 \leq i \leq 4$) of Figure 2.3 may be trivial.

Theorem 2.52 [46] *For a connected graph G with m edges, we have $rc(G) \neq m - 1$; and $rc(G) = m - 2$ if and only if G is a 5-cycle or belongs to one of four graph classes \mathcal{G}_i 's ($1 \leq i \leq 4$) shown in Figure 2.3.*

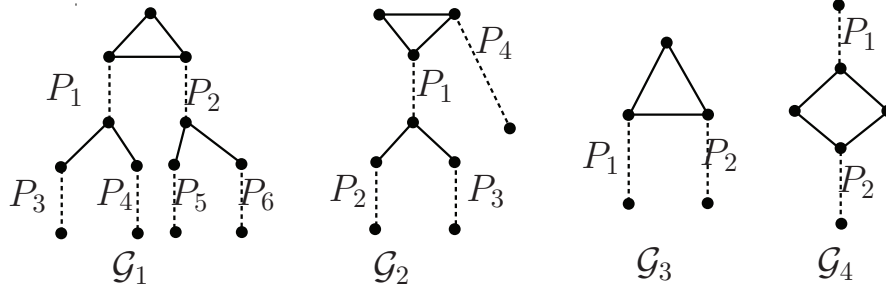


Figure 2.3 The figure for the four graph classes.

We now introduce two graph classes. Let C be the cycle of a unicyclic graph G , $V(C) = \{v_1, \dots, v_k\}$ and $\mathcal{T}_G = \{T_i : 1 \leq i \leq k\}$ where T_i is the unique tree containing vertex v_i in subgraph $G \setminus E(C)$. We say T_i and T_j are *adjacent* (*nonadjacent*) if v_i and v_j are adjacent (nonadjacent) in cycle C . Then let

$\mathcal{G}_5 = \{G : G \text{ is a unicyclic graph, } k = 3, \mathcal{T}_G \text{ contains at most two nontrivial elements}\},$

$\mathcal{G}_6 = \{G : G \text{ is a unicyclic graph, } k = 4, \mathcal{T}_G \text{ contains two nonadjacent trivial elements and the other two (nonadjacent) elements are paths}\}.$

Theorem 2.53 [47] *For a connected graph G with m edges, $\text{src}(G) \neq m-1$; $\text{src}(G) = m-2$ if and only if G is a 5-cycle or belongs to one of the \mathcal{G}_i 's ($i = 5, 6$).*

By Proposition 2.1, Theorems 2.52 and 2.53, we investigated the graphs with $\text{rc}(G) \geq m-2$ ($\text{src}(G) \geq m-2$). Furthermore, we have the following interesting problem.

Problem 2.54 *Give a sufficient condition to guarantee $\text{rc}(G) \geq \alpha m$ ($\text{src}(G) \geq \alpha m$), where $0 < \alpha < 1$.*

2.5 For graph operations

Products of graphs occur naturally in discrete mathematics as tools in combinatorial constructions, they give rise to important classes of graphs and deep structural problems. The extensive literature on products that has evolved over the years presents a wealth of profound and beautiful results [32]. In [46], Li and Sun obtained some results on the rainbow connection numbers of products of graphs, including Cartesian product, composition (lexicographic product), union of graphs, etc. Actually, we know that the line graph of a graph is also a graph operation.

We first introduce the rainbow connection number of the Cartesian product of some graphs. The *Cartesian product* of graphs G and H is the graph $G \square H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1 u_2 \in E(G)$ and $v_1 = v_2$, or $v_1 v_2 \in E(H)$ and $u_1 = u_2$. The *strong product* of G and H is the graph $G \boxtimes H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs

$(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2$, or $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$. By definition, the graph $G \square H$ is the spanning subgraph of the graph $G \boxtimes H$. By using the definition and structure of Cartesian product, Li and Sun derived the following.

Theorem 2.55 [46] *Let $G^* = G_1 \square G_2 \square \cdots \square G_k$ ($k \geq 2$), where each G_i ($1 \leq i \leq k$) is connected. Then we have*

$$rc(G^*) \leq \sum_{i=1}^k rc(G_i).$$

Moreover, if $diam(G_i) = rc(G_i)$ for each G_i , then the equality holds.

We know that there are infinity many graph with $diam(G) = rc(G)$, such as the unit interval graphs shown in Theorem 2.41. The following problem could be interesting but maybe difficult:

Problem 2.56 *Characterize the graphs G with $rc(G) = diam(G)$, or give some sufficient conditions to guarantee that $rc(G) = diam(G)$.*

Similar problem for $src(G)$ can be considered.

Let $\overline{G^*} = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_k$ ($k \geq 2$), where each G_i ($1 \leq i \leq k$) is connected. Since the Cartesian product of any two graphs is a spanning subgraph of their strong product, G^* is the spanning subgraph of $\overline{G^*}$, then we have the following result.

Corollary 2.57 [46] *Let $\overline{G^*} = G_1 \boxtimes G_2 \boxtimes \cdots \boxtimes G_k$ ($k \geq 2$), where each G_i ($1 \leq i \leq k$) is connected. Then we have*

$$rc(\overline{G^*}) \leq \sum_{i=1}^k rc(G_i).$$

For $i = 1, 2, \dots, r$, let $m_i \geq 2$ be given integers. Consider the graph G whose vertices are the r -tuples $b_1b_2 \cdots b_r$ with $b_i \in \{0, 1, \dots, m_i - 1\}$, and let two vertices be adjacent if the corresponding tuples differ in precisely one place. Such a graph is called a *Hamming graph*. Clearly, a graph G is a Hamming graph if and only if it can be written in the form $G = K_{m_1} \square K_{m_2} \cdots \square K_{m_r}$ for some $r \geq 1$, where $m_i \geq 2$ for all i . So we call G a Hamming graph with r factors. A Hamming graph is a *hypercube* (or *r -cube*) [28], denoted by Q_r , if and only $m_i = 2$ for all i . The concept of Hamming graph is useful in communication networks [32].

Corollary 2.58 [46] *If G is a Hamming graph with r factors, then $rc(G) = r$. In particular, $rc(Q_r) = r$.*

The *composition* (lexicographic product) of two graphs G and H is the simple graph $G[H]$ with vertex set $V(G) \times V(H)$ in which (u, v) is adjacent to (u', v') if and only if either

$uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. By definition, $G[H]$ can be obtained from G by substituting a copy H_v of H for every vertex v of G and by joining all vertices of H_v with all vertices of H_u if $uv \in E(G)$. Note that $G[H]$ is connected if and only if G is connected. By definition, it is easy to show: If G is complete, then $\text{diam}(G[H]) = 1$ if H is complete (as now $G[H]$ is complete), $\text{diam}(G[H]) = 2$ if H is not complete; If G is not complete, then $\text{diam}(G[H]) = \text{diam}(G)$. Then we have the following result.

Theorem 2.59 [46] *If G and H are two graphs and G is connected, then we have*

(1) *if H is complete, then*

$$rc(G[H]) \leq rc(G).$$

In particular, if $\text{diam}(G) = rc(G)$, then $rc(G[H]) = rc(G)$.

(2) *if H is not complete, then*

$$rc(G[H]) \leq rc(G) + 1.$$

In particular, if $\text{diam}(G) = rc(G)$, then $\text{diam}(G[H]) = 2$ if G is complete and $rc(G) \leq \text{diam}(G[H]) \leq rc(G) + 1$ if G is not complete.

In [46], Li and Sun also investigated other graph operations, such as the union of graphs which we will not introduce here.

2.6 An upper bound for strong rainbow connection number

The topic of rainbow connection number is fairly interesting and recently a series papers have been published about it. The strong rainbow connection number is also interesting, and by definition, the investigation of it is more challenging than that of the rainbow connection number. However, there are very few papers that have been published about it. In [12], Chartrand, Johns, McKeon and Zhang determined the precise strong rainbow connection numbers for some special graph classes including trees, complete graphs, wheels and complete bipartite (multipartite) graphs as shown in Subsection 2.1. However, for a general graph G , it is almost impossible to give the precise value for $\text{src}(G)$, so we aim to give upper bounds for it according to some graph parameters. Li and Sun [47] derived a sharp upper bound for $\text{src}(G)$ according to the number of edge-disjoint triangles (if exist) in a graph G , and give a necessary and sufficient condition for the sharpness. We need to introduce a new graph class.

Recall that a *block* of a connected graph G is a maximal connected subgraph without a cut vertex. Thus, every block of G is either a maximal 2-connected subgraph or a bridge. We now introduce a new graph class. For a connected graph G , we say $G \in \overline{\mathcal{G}}_t$, if it satisfies the following conditions:

C_1 . Each block of G is a bridge or a triangle;

C_2 . G contains exactly t triangles;

C_3 . Each triangle contains at least one vertex of degree two in G .

By the definition, each graph $G \in \overline{\mathcal{G}}_t$ is formed by (edge-disjoint) triangles and paths (may be trivial), these triangles and paths fit together in a treelike structure, and G contains no cycles but the t (edge-disjoint) triangles. For example, see Figure 2.4, here $t = 2$, and

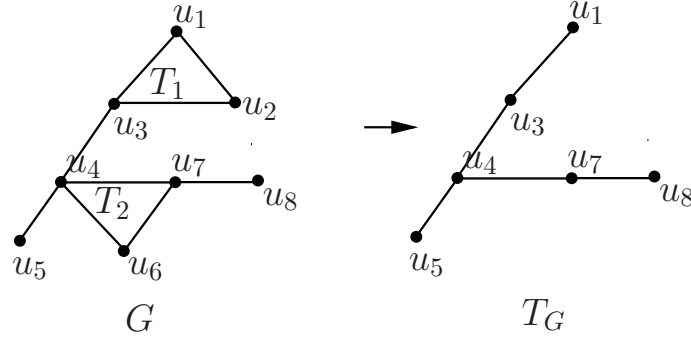


Figure 2.4 An example of $G \in \overline{\mathcal{G}}_t$ with $t = 2$.

u_1 , u_2 and u_6 are vertices of degree 2 in G . If a tree is obtained from a graph $G \in \overline{\mathcal{G}}_t$ by deleting one vertex of degree 2 from each triangle, then we call this tree is a D_2 -tree of G , denoted by T_G . For example, in Figure 2.4, T_G is a D_2 -tree of G . Clearly, the D_2 -tree is not unique, since in this example, we can obtain another D_2 -tree by deleting the vertex u_1 instead of u_2 . On the other hand, we can say that any element of $\overline{\mathcal{G}}_t$ can be obtained from a tree by adding t new vertices of degree 2. It is easy to show that the number of edges of T_G is $m - 2t$ where m is the number of edges of G .

They derived the following result.

Theorem 2.60 [47] *If G is a graph with m edges and t edge-disjoint triangles, then*

$$src(G) \leq m - 2t,$$

the equality holds if and only if $G \in \overline{\mathcal{G}}_t$.

In [1], Ahadi and Dehghan also derived an upper bound for strongly regular graph: if G is an $SRG(n, r, \lambda, \mu)$, then $src(G) \leq \lceil (e(4\mu r - 4\mu\lambda - 6\mu + 1))^{\frac{1}{\mu}} \rceil$. However, we do not know whether it is sharp.

Unlike rainbow connection number, which is a monotone graph property (adding edges never increases the rainbow connection number), this is not the case for the strong rainbow connection number (see Figure 2.1 for an example). The investigation of strong rainbow connection number is much harder than that of rainbow connection number. Chakraborty, Fischer, Matsliah and Yuster gave the following conjecture.

Conjecture 2.61 [10] *If G is a connected graph with minimum degree at least en , then it has a bounded strong rainbow connection number.*

3 Rainbow k -connectivity

In this section, we survey the results on rainbow k -connectivity. By its definition, we know that it is difficult to derive exact value or a nice upper bound of the rainbow k -connectivity for a general graph. Chartrand, Johns, McKeon and Zhang [13] did some basic research on the rainbow k -connectivity of two special graph classes. They first studied the rainbow k -connectivity of the complete graph K_n for various pairs k, n of integers, and derived the following result:

Theorem 3.1 [13] *For every integer $k \geq 2$, there exists an integer $f(k)$ such that if $n \geq f(k)$, then $rc_k(K_n) = 2$.*

They obtained an upper bound $(k+1)^2$ for $f(k)$, namely $f(k) \leq (k+1)^2$. Li and Sun [44] continued their investigation, and the following result is derived:

Theorem 3.2 [44] *For every integer $k \geq 2$, there exists an integer $f(k) = ck^{\frac{3}{2}} + C(k)$ where c is a constant and $C(k) = o(k^{\frac{3}{2}})$ such that if $n \geq f(k)$, then $rc_k(K_n) = 2$.*

From Theorem 3.2, we can obtain an upper bound $ck^{\frac{3}{2}} + C(k)$ for $f(k)$, where c is a constant and $C(k) = o(k^{\frac{3}{2}})$, that is, they improved the upper bound of $f(k)$ from $O(k^2)$ to $O(k^{\frac{3}{2}})$, a considerable improvement. Dellamonica, Magnant and Martin [21] got the best possible upper bound $2k$, which is linear in k (see Theorem 3.3). However, the proof of Theorem 3.2 is more structural or constructive, and informative. In the argument of [21], Dellamonica, Magnant and Martin put forward a new concept, the *rainbow (k, l) -connectivity*.

Given an edge-colored simple graph G , let $l \leq k$ be integers. Suppose the edges of G are k -colored. For $a, b \in V(G)$, denote by $p(a, b)$ the maximum number of internally disjoint rainbow paths of length l having endpoints a and b . The *rainbow (k, l) -connectivity* of G is the minimum $p(a, b)$ among all distinct $a, b \in V(G)$. Note that this new defined *rainbow (k, l) -connectivity* computes the number of internally disjoint paths with the *same* length l (this is distinct from the rainbow k -connectivity which, as mentioned above, computes the number of colors); and by definition, for different edge-colorings, the values of *rainbow (k, l) -connectivity* could be different.

From Theorem 3.1, we know that for any r , there exists an explicit 2-coloring of K_r in which the number of bi-chromatic paths of length 2 between any pair of vertices is at least $\lfloor \sqrt{r} - 1 \rfloor$. Using the above definition, it is a statement about the rainbow (2,2)-connectivity of a given 2-coloring of the edges of K_r . In [21], Dellamonica, Magnant and Martin greatly improve and generalize the above lower bound for graphs of sufficiently large order by providing a different constructive coloring. Their construction attains asymptotically the maximum rainbow connectivity possible.

Theorem 3.3 [21] *For any $k \geq 2$ and $r \geq r_0 = r_0(k)$ there exists an explicit k -coloring of the edges of K_r having rainbow $(k, 2)$ -connectivity*

$$(\frac{k-1}{k} - o(1))r.$$

More generally, They considered the problem of finding longer rainbow paths.

Theorem 3.4 [21] *For any $3 \leq l \leq k$, there exists $r_0 = r_0(k)$ such that for every $r \geq r_0$, there is an explicit k -coloring of the edges of K_r having rainbow $(k, 2)$ -connectivity*

$$(1 - o(1))\frac{r}{l-1}.$$

This result is also asymptotically best possible, since any collection of internally disjoint paths of length l can contain at most $\frac{r}{l-1}$ paths. Their proof employed a very recent breakthrough due to Bourgain [8, 51], which consists of a powerful explicit extractor. Roughly speaking, an (explicit) extractor is a polynomial time algorithm used to convert some special probability distributions into uniform distributions. See [54] for a good but somewhat outdated survey on extractors.)

Chartrand, Johns, McKeon and Zhang [13] also investigated the rainbow k -connectivity of r -regular complete bipartite graphs for some pairs k, r of integers with $2 \leq k \leq r$, and they showed:

Theorem 3.5 [13] *For every integer $k \geq 2$, there exists an integer r such that $rc_k(K_{r,r}) = 3$.*

However, they could not show a similar result for complete graphs, and therefore they left an open question: For every integer $k \geq 2$, determine an integer (function) $g(k)$, for which $rc_k(K_{r,r}) = 3$ for every integer $r \geq g(k)$, that is, the rainbow k -connectivity of the complete bipartite graph $K_{r,r}$ is essentially 3. In [45], Li and Sun solved this question using a similar but more complicated method to that of Theorem 3.5, and they proved:

Theorem 3.6 [45] *For every integer $k \geq 2$, there exists an integer $g(k) = 2k \lceil \frac{k}{2} \rceil$ such that $rc_k(K_{r,r}) = 3$ for any $r \geq g(k)$.*

More generally, we have the following question.

Question 3.7 [13] *Does the following hold? For each integer $k \geq 2$, there exists an integer $h(k)$ such that for every two integers s and t with $h(k) \leq s \leq t$, we have $rc_k(K_{s,t}) = 3$.*

Recently, He and Liang [29] investigated the rainbow k -connectivity in the setting of random graphs. They determined a sharp threshold function for the property $rc_k(G(n, p)) \leq d$ for every fixed integer $d \geq 2$. This substantially generalizes a result due to Caro, Lev, Roditty, Tuza and Yuster (see Theorem 2.46). Their main result is as follows.

Theorem 3.8 [29] *Let $d \geq 2$ be a fixed integer and $k = k(n) \leq O(\log n)$. Then $p = \frac{(\log n)^{1/d}}{n^{(d-1)/d}}$ is a sharp threshold function for the property $rc_k(G(n, p)) \leq d$.*

The key ingredient of their proof is the following result: With probability at least $1 - n^{-\Omega(1)}$, every two different vertices of $G(n, \frac{C(\log n)^{1/d}}{n^{(d-1)/d}})$ are connected by at least $2^{10d} c_0 \log n$ internally disjoint paths of length exactly d .

So far, results on the rainbow k -connectivity are just those for two special graph classes, and a sharp threshold function for the property $rc_k(G(n, p)) \leq d$ where $d \geq 2$ is a fixed integer and $k = k(n) \leq O(\log n)$. Clearly, there are a lot of things one can do further on this concept. As mentioned above, it is difficult to derive the precise value or a nice upper bound for $rc_k(G)$ of a κ -connected graph G , where $2 \leq k \leq \kappa$. So one may consider the following problem.

Problem 3.9 *Derive a sharp upper bound for $rc_2(G)$, where G is a κ -connected graph with $\kappa \geq 2$. Does $rc_2(G) \leq \alpha n$, where $0 < \alpha < 1$ is independent of n ?*

4 k -rainbow index

The k -rainbow coloring as defined above, is another generalization of the rainbow coloring. In [15], Chartrand, Okamoto and Zhang did some basic research on this topic. There is a rather simple upper bound for $rx_k(G)$ in terms of the order of G , regardless the value of k .

Proposition 4.1 [15] *Let G be a nontrivial connected graph of order $n \geq 3$. For each integer k with $3 \leq k \leq n - 1$, $rx_k(G) \leq n - 1$ while $rx_n(G) = n - 1$.*

There is a class of graphs of order $n \geq 3$ whose k -rainbow index attains the upper bound of Proposition 4.1, it is an immediate consequence of Observation 2.11.

Proposition 4.2 [15] *Let T be a tree of order $n \geq 3$. For each integer k with $3 \leq k \leq n$, $rx_k(T) = n - 1$.*

There is also a rather obvious lower bound for the k -rainbow index of a connected graph G of order n , where $3 \leq k \leq n$. The *Steiner distance* $d(S)$ of a set S of vertices in G is the minimum size of a tree in G containing S . Such a tree is called a *Steiner S -tree* or simply a *Steiner tree*. The k -Steiner diameter, say $sdiam_k(G)$ of G is the maximum Steiner distance of S among all sets S with k vertices in G . Thus if $k = 2$ and $S = \{u, v\}$, then $d(S) = d(u, v)$ and the 2-Steiner diameter $sdiam_2(G) = diam(G)$. The k -Steiner diameter then provides a lower bound for the k -rainbow index of G : for every connected graph G of order $n \geq 3$ and each integer k with $3 \leq k \leq n$, $rx_k(G) \geq sdiam_k(G) \geq k - 1$.

Theorem 4.3 [15] *If G is a unicyclic graph of order $n \geq 3$ and girth $g \geq 3$, then*

$$rx_k(G) = \begin{cases} n-2 & \text{if } k=3 \text{ and } g \geq 4, \\ n-1 & \text{if } g=3 \text{ or } 4 \leq k \leq n. \end{cases}$$

The investigation of $rx_k(G)$ for a general k and a general graph G is rather difficult. So one may consider $rx_k(G)$ for a special graph class, or, for a general graph and small k , such as $k=3$.

Problem 4.4 *Derive a sharp upper bound for $rx_3(G)$.*

There is also a generalization of the k -rainbow index, say (k, ℓ) -rainbow index $rx_{k,\ell}$, of G which is mentioned in [15] and we will not introduce it here.

5 Rainbow vertex-connection number

The above several parameters are all defined on edge-colored graphs. Here we will introduce a new graph parameter which is defined on vertex-colored graphs. It is, as mentioned above, a vertex-version of the rainbow connection number. Krivelevich and Yuster [36] put forward this new concept and proved a theorem analogous to Theorem 2.17.

Theorem 5.1 [36] *A connected graph G with n vertices has $rvc(G) < \frac{11n}{\delta(G)}$.*

The argument of this theorem used the concept of k -strong two-step dominating sets. They proved that if H is a connected graph with n vertices and minimum degree δ , then it contains a $\frac{\delta}{2}$ -strong two-step dominating set S whose size is at most $\frac{2n}{\delta+2}$. Then they derived an edge-coloring for G according to its connected $\frac{\delta}{2}$ -strong two-step dominating set. And they showed that, with positive probability, their coloring yields a rainbow connected graph by the Lovász Local Lemma (see [3]).

Motivated by the method of Theorem 5.1, Li and Shi derived an improved result.

Theorem 5.2 [40] *A connected graph G of order n with minimum degree δ has $rvc(G) \leq 3n/(\delta+1) + 5$ for $\delta \geq \sqrt{n-1} - 1, n \geq 290$, while $rvc(G) \leq 4n/(\delta+1) + 5$ for $16 \leq \delta \leq \sqrt{n-1} - 2$, $rvc(G) \leq 4n/(\delta+1) + C(\delta)$ for $6 \leq \delta \leq 16$ where $C(\delta) = e^{\frac{3 \log(\delta^3 + 2\delta^2 + 3) - 3(\log 3 - 1)}{\delta - 3}} - 2$, $rvc(G) \leq n/2 - 2$ for $\delta = 5$, $rvc(G) \leq 3n/5 - 8/5$ for $\delta = 4$, $rvc(G) \leq 3n/4 - 2$ for $\delta = 3$. Moreover, an example shows that when $\delta \geq \sqrt{n-1} - 1$, and $\delta = 3, 4, 5$ the bounds are seen to be tight up to additive factors.*

Motivated by the method of Theorem 5.1, Dong and Li [22] also proved a theorem analogous to Theorem 2.21 for the rainbow vertex-connection version according to the degree sum condition σ_2 , which is stated as the following theorem.

Theorem 5.3 [22] *For a connected graph G of order n , $rc(G) \leq 8\frac{n-2}{\sigma_2+2} + 10$ for $2 \leq \sigma_2 \leq 6$ and $\sigma_2 \geq 28$, while for $7 \leq \sigma_2 \leq 8$ and $16 \leq \sigma_2 \leq 27$, $rc(G) \leq \frac{10n-16}{\sigma_2+2} + 10$, and for $9 \leq \sigma_2 \leq 15$, $rc(G) \leq \frac{10n-16}{\sigma_2+2} + A(\sigma_2)$, where $A(\sigma_2) = 63, 41, 27, 20, 16, 13, 11$, respectively.*

Note that by the definition of σ_2 , we know $\sigma_2 \geq 2\delta$, so we have $8\frac{n-2}{\sigma_2+2} + 10 \leq \frac{4(n-2)}{\delta+1} + 10$, and hence the bound of Theorem 5.3 is an improvement to that of Theorem 5.2 in the case $16 \leq \delta \leq \sqrt{n-1} - 2$.

6 Algorithms and computational complexity

At the end of [9], Caro, Lev, Roditty, Tuza and Yuster gave two conjectures (see Conjecture 4.1 and Conjecture 4.2 in [9]) on the complexity of determining the rainbow connection numbers of graphs. Chakraborty, Fischer, Matsliah and Yuster [10] solved these two conjectures by the following theorem.

Theorem 6.1 [10] *Given a graph G , deciding if $rc(G) = 2$ is NP-Complete. In particular, computing $rc(G)$ is NP-Hard.*

Chakraborty, Fischer, Matsliah and Yuster divided the proof of Theorem 6.1 into three steps: the first step is showing the computational equivalence of the problem of *rainbow connection number 2*, that asks for a red-blue edge coloring in which all vertex pairs have a rainbow path connecting them, to the problem of *subset rainbow connection number 2*, that asks for a red-blue coloring in which every pair of vertices in a given subset of pairs has a rainbow path connecting them. In the second step, they reduced the problem of *extending to rainbow connection number 2*, asking whether a given partial red-blue coloring can be completed to obtain a rainbow connected graph, to the problem of subset rainbow connection number 2. Finally, the proof of Theorem 6.1 is completed by reducing 3-SAT to the problem of extending to rainbow connection number 2.

Chakraborty, Fischer, Matsliah and Yuster [10] also raised the following problem.

Problem 6.2 [10] *Suppose that we are given a graph G for which we are told that $rc(G) = 2$. Can we rainbow-color it in polynomial time with $o(n)$ colors?*

For the usual coloring problem, this version has been well studied. It is known that if a graph is 3-colorable (in the usual sense), then there is a polynomial time algorithm that colors it with $\tilde{O}(n^{3/14})$ colors [5].

Suppose we are given an edge coloring of the graph. Is it then easier to verify whether the colored graph is rainbow connected? Clearly, if the number of colors is constant then this problem becomes easy. However, if the coloring is arbitrary (with an unbounded number of colors), the problem becomes NP-Complete.

Theorem 6.3 [10] *The following problem is NP-Complete: Given an edge-colored graph G , check whether the given coloring makes G rainbow connected.*

For the proof of Theorem 6.3, Chakraborty, Fischer, Matsliah and Yuster first showed that the $s - t$ version of the problem is NP-Complete. That is, given two vertices s and t of an edge-colored graph, decide whether there is a rainbow path connecting them. Then they reduced the problem of Theorem 6.3 from it.

More generally it has been shown in [37], that for any fixed $k \geq 2$, deciding if $rc(G) = k$ is NP-complete.

In [10], Chakraborty, Fischer, Matsliah and Yuster also derived some positive algorithmic results. Parts of the following two results were shown in Theorems 2.16 and 2.48. They proved:

Theorem 6.4 [10] *For every $\epsilon > 0$ there is a constant $C = C(\epsilon)$ such that if G is a connected graph with n vertices and minimum degree at least ϵn , then $rc(G) \leq C$. Furthermore, there is a polynomial time algorithm that constructs a corresponding coloring for a fixed ϵ .*

As mentioned above, Theorem 6.4 is based upon a modified degree-form version of Szemerédi's Regularity Lemma that they proved and that may be useful in other applications. From their algorithm it is also not hard to find a probabilistic polynomial time algorithm for finding this coloring with high probability (using on the way the algorithmic version of the Regularity Lemma from [2] or [26]).

Theorem 6.5 [10] *If G is an n -vertex graph with diameter 2 and minimum degree at least $8 \log n$, then $rc(G) \leq 3$. Furthermore, such a coloring is given with high probability by a uniformly random 3-edge-coloring of the graph G , and can also be found by a polynomial time deterministic algorithm.*

As mentioned, He and Liang [29] investigated the rainbow k -connectivity in the setting of random graphs. They also investigated rainbow k -connectivity from the algorithmic point of view. The NP-hardness of determining $rc(G)$ was shown by Chakraborty et al. as shown above. They showed that the problem (even the search version) becomes easy in random graphs.

Theorem 6.6 [29] *For any constant $\epsilon \in [0, 1)$, $p = n^{-\epsilon(1 \pm o(1))}$ and $k \leq O(\log n)$, there is a randomized polynomial time algorithm that, with probability $1 - o(1)$, makes $G(n, p)$ rainbow- k -connected using at most one more than the optimal number of colors.*

Since almost all natural edge probability functions p encountered in various scenarios have such form, their result is quite strong. Note that $G(n, n^{-\epsilon})$ is almost surely disconnected

when $\epsilon > 1$ [24], which makes the problem become trivial. Therefore they ignored these cases.

Recall that the values of rainbow (k, l) -connectivity may be different for distinct edge-colorings. In [21], Dellamonica, Magnant and Martin derived that a random k -coloring of a sufficiently large complete graph has asymptotically optimal rainbow (k, l) -connectivity (see Theorems 3.3 and 3.4). They obtained an *explicit* edge-coloring. By explicit, we mean that they gave a polynomial time algorithm to compute such an edge-coloring.

Recently, the computational complexity of rainbow vertex-connection numbers has been determined by Chen, Li and Shi [18].

Motivated by the proofs of Theorems 6.1 and 6.3, they derived two corresponding results to the rainbow vertex-connection.

Theorem 6.7 [18] *Given a graph G , deciding if $rvc(G) = 2$ is NP-Complete. In particular, computing $rvc(G)$ is NP-Hard.*

Theorem 6.8 [18] *The following problem is NP-Complete: given a vertex-colored graph G , check whether the given coloring makes G rainbow vertex-connected.*

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